Notes on Convergence

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Abstract

This document describes a series of intuitions and techniques to *very quickly* gauge the convergence or divergence of most simple series encountered in a Calculus-II or equivalent course.

1 Disclaimer

Due to the relative lack of both breadth and depth, this document is not meant to replace or displace other study methods. I make no guarantees for self sufficiency; use this as a supplement or a cheat-sheet, but nothing more.

Contents

2 All tests

A quite useful mnemonic to remember the name of all the tests is the "word" **PDARRLING.** As follows,

- **P** – $P-$series
- **D** – Direct comparison
- **A** – Alternating series
- **R** – Ratio test
- **R** – Root test
- **L** – Limit comparison
- **I** – Integral test
- **N** – $n-$th term test
- **G** – Geometric series criterion.

When asked to *rigorously* identify which test to use for a given series, apply the following steps:

1. Is it a special series such as a $P-$series or a geometric series? Use their corresponding tests.
2. Mentally check the $n-$th term test. Actually cite if the $n-$th term test shows the series diverges, but continue otherwise.
3. Does the series look a lot like a different series you already know converges or diverges? Apply the limit comparison test.
    - If you have to rely on the fact that some class of functions dominates another or there is some mixing of classes (such as factorials, exponential or polynomials), apply direct comparison in some capacity.
4. Does the series have a really *long* expression raised to the $n-$th power? Apply the root test.
5. Does the series have a factorial and/or an exponential and polynomial? Apply the ratio test.
6. Does the series have a $(-1)^n$ in some way? Apply the alternating series test.
7. Apply the integral test.

Note that you should carry out the first six steps *immediately* for a simple series; the majority of the time, it should be blatantly obvious which test(s) you ought to use for a series.

Symbolically, we can adhere to the following table:

Situation	Test	Condition(s)
Num. > Denom.	$n-$th term	If $\lim_{n\to\infty} a_n \neq 0$, diverges.
$\sum_{n=k}^{\infty} \frac{1}{n^p}$	$P-$series	If $p \leq 1$, diverges. If $p > 1$, converges.
$\sum_{n=k}^{\infty} r^n$	Geometric series	If $\lvert r\rvert \geq 1$, diverges. If $\lvert r\rvert < 1$, converges.
Lookalike	Limit comparison	If $\lim_{n\to\infty} \frac{a_n}{b_n} \notin \mathbb{R}$, diverges. If $\lim_{n\to\infty} \frac{a_n}{b_n} \in \mathbb{R}$, converges.
$\sum_{n=k}^{\infty} (a_n)^n$	Root test	If $\lim_{n\to\infty} a_n \geq 1$, diverges. If $\lim_{n\to\infty} a_n < 1$, converges.
"Mixed classes"	Ratio test	If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$, diverges If $\frac{a_{n+1}}{a_n} < 1$, converges. If $\frac{a_{n+1}}{a_n} = 1$, inconclusive.
$\sum_{n=k}^{\infty}(-1)^n a_n$	Alternating series	If $\lvert a_n\rvert$ is monotonically decreasing *and* $\lim_{n\to\infty} a_n = 0$, converge.
No tests left	Integral test	If $f(n) = a_n$, given that $f(n)$ is positive *and* monotonically decreasing, if $\int_k^{\infty} f(x)\ dx$ diverges, diverges. If $\int_k^{\infty} f(x)\ dx$ converges, converges.

To gauge if a given series $\sum_{n=k}^{\infty} a_n$ converges absolutely, simply see if $\sum_{n=k}^{\infty} |a_n|$ converges. A series where $\sum_{n=k}^{\infty} |a_n|$ converges is said to converge absolutely. If $\sum_{n=k}^{\infty} |a_n|$ diverges but $\sum_{n=k}^{\infty} a_n$ converges, the series is said to be conditionally convergent.

3 Weird geometric series

Certain series are unnecessarily *tricky* – I don't mean this to mean that numerous tests are necessary or that the algebraic work for the test is really involved – I mean this to mean that you are expected to use techniques that may not be formally taught in order to get an answer. I will enumerate some here:

Example 1: MA 162 EXAM 2 Q12

Does the series

$$\sum_{n=1}^{\infty} \frac{1+3^n}{7^n}$$

converge? If so, what is its sum?

Outline: Clearly a question that asks you for a value has to be either a geometric series or a telescoping series, and the 3^n and 7^n suggest it to be the former. Unfortunately, there does not exist a common ratio, so it's not a normal geometric series. The trick is to split the expression into two.

1. Split $\sum_{n=1}^{\infty} \frac{1+3^n}{7^n}$ into two sums.
2. Compute the value of the two sums.
3. Add the two sums together.
4. Answer the question.

Example 2: Paul's Online Math Notes; Special Series

Determine if the following series [converges or diverges.] If [it converges] give the value of the series.

$$\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1}.$$

Outline: This doesn't really look like a geometric series, but actually is one. The trick is to extract the ns by exponent rules.

1. "Factor out", by exponent rules, 9^{-n} to get 9^{-n} and 9^2. Repeat this process with 4^n and 4^1.
2. Rewrite the expression and put 9^2 right next to 4^1, put 9^{-n} and 4^n next to each other.
3. Note that $4^1 \times 9^2$ is just a constant which can be "factored out" of the sum. This leaves you with just $4^n \cdot 9^{-n}$.
4. Rewrite $4^n \cdot 9^{-n}$ as $\left(\frac{4^n}{9^n}\right)$.
5. Conclude.

Example 3: Paul's Online Math Notes; Special Series

Determine if the following series [converges or diverges.] If [it converges] give the value of the series.

$$\sum_{n=0}^{\infty} \frac{(-4)^{3n}}{5^{n-1}}.$$

Outline: The trick is that for a given geometric series, we want it in terms of r^n. Unfortunately the $(-4)^{3n}$ gets in the way with it.

1. Rewrite $(-4)^{3n}$ as $\left((-4)^3\right)^n$.
2. Simplify $\left((-4)^3\right)^n$ to $(-64)^n$.
3. Rewrite 5^{n-1} as $5^{-1} \cdot 5^n$, then "factor out" 5^{-1}.
4. Conclude.

4 Quick & accurate guessing

> The most efficient way to solve a problem is to simply guess the correct answer.
>
> AJ Shaka, PhD.

By taking advantage of a few theorems, most convoluted expressions can be reduced incredibly quickly. Almost every scary looking series can be reduced to something very easily workable.

Theorem 1: Function growth

Regardless of base, factorials dominate exponential functions dominates polynomials dominates constants.

Polynomials dominate logarithms, but logarithms themselves do not dominate constants. Specifically linear polynomials do not dominate constants, either.

$$n! \gg a^n \gg n^a \gg a.$$

This can be proved by induction and a bunch of other ways, but I won't bother with it here.

This theorem is incredibly useful as it enables us not only to simplify complicated expressions down to at most *two expressions*, but it can also enable us to form a conclusion immediately after. Of course, once we have our conclusion, we can apply the tests to verify our answer. The rough procedure is as follows:

1. Use limit domination to simplify numerators and denominators.
2. Use limit domination to compare the numerator and denominator.
3. Conclude.

Example 4: MA 162 EXAM 3 2018 Q4*

Discern if the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2}.$$

Solution: Note that the numerator of this fraction is an exponential function and the denominator of this fraction is a polynomial. Since the numerator dominates the denominator, the series diverges. □

Of course, this isn't a rigorous proof. To actually show that the series converges or diverges, use the ratio test.

Example 5: MA 162 EXAM 3 2018 Q4*

Discern if the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{1}{2^n(1+n)}.$$

Solution: Note that exponential functions dominate polynomials, we can rewrite the denominator as simply 2^n which gives us $\sum_{n=1}^{\infty} \frac{1}{2^n}$. Then, by observing that the denominator dominates the numerator, the series converges. □

Example 6: MA 162 EXAM 3 2018 Q4*

Discern if the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{n^2}{3^n}.$$

Solution: Note that the denominator dominates the numerator, so the series converges. □

Example 7: MA 162 FINAL 2017 Q18*

iscern if the following series converges or diverges:

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}.$$

Solution: Polynomials dominate logarithms, so the denominator can be rerwritten as n. The series is then $\sum_{n=2}^{\infty} \frac{1}{n}$, which diverges as the harmonic series. □

Example 8: MA 162 EXAM 3 2017 Q1*

Discern if the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{n}{n^2 + 7}.$$

Solution: Note that polynomials dominate constants, so we can rewrite the denominator as n^2. The series is then $\sum_{n=1}^{\infty} \frac{n}{n^2}$, which simplifies to $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges as the harmonic series. □

Example 9: MA 162 EXAM 3 2017 Q3*

Discern if the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{3n^2}{n^3 + 1}.$$

Solution: Rewrite the denominator as n^3. The series is then $\sum_{n=1}^{\infty} \frac{3n^2}{n^3}$, which simplifies to $3\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges as the harmonic series. □

Example 10: MA 162 EXAM 3 2017 Q3*

Discern if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{3n^2}{(n^3 + 1)^{4/3}}$$

Solution: Rewrite the denominator as $\left(n^3\right)^{4/3}$, which can then be simplified to n^4. The series is then $\sum_{n=1}^{\infty} \frac{3n^2}{n^4}$, which converges after simplification. □