District Meet 2019 March Notes

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2019 March 18

These are my **unofficial** solutions for the March 2019 district meet, which occurred on the 14th of March. These solutions contain my thoughts and motivation for each and every problem. Additionally, some questions contain my own personal commentary relating to compeition day. As always, questions are given in sequential order.

Contents

1 All problems

ARITHMETIC:

1. In 2000's so far, 2000-2018, the mean rainfall for March in Sacramento is 2.76 inches. How many inches of rain would Sacramento have to get in March of 2019 to raise the overall mean for the 2000's to 3 inches?

2. Bob went to a hamburger restaurant on a night where burgers were 50% off. He ordered a burger and sides. (The sides were priced at regular prices.) He paid 5% tax on his meal, which amounted to 62 cents. He decided to tip the server 15% of the amount he would have paid for the meal (excluding tax) if burgers were at full price. His tip was \$2.43. What is the regular price of a burger?

   Note: No rounding was necessary in computing either the tax or the tip.

TRIG/ANALYSIS:

1. Define
$$T_n = 1^7 + 2^7 + 3^7 + \ldots + n^7$$
   for all natural numbers n. Write the following expression in simplest form.
$$\log_{\sqrt{7}}(T_{343} - T_{342})$$

2. Three different math classes at a local high school each send three students on a trip to see a performance of Pi – The Musical. They are randomly assigned seats in the front row, which conveniently has 9 chairs. What is the probability that students from each class will sit together in a block of 3 consecutive seats?

GEOMETRY:

1. A triangle with sides 5, 12, and 13 is inscribed in a circle. What is the circumference of the circle?

2. Given: Quadrilateral $ABCD$. $\overline{BD} \perp \overline{AD}$, $\overline{BD} \perp \overline{BC}$, $BD = 8$, $AB = 17$, $CD = 10$. What is the area of Quadrilateral $ABCD$?

ALGEBRA:

1. Olen Muks invests a certain amount dollars in Tesla, Inc stock. To his delight, his investment increased by 10%. Being the cautious investor, he then takes \$130 of his investment out of the stock market and hides it under his mattress. The amount of his investment remaining in Tesla stock then decreases by 10%. Olen then pulls all of his money out of the stock and puts it under his mattress. The amount of money under his mattress is 3% more than what he originally invested. How much did Olen Muks originally invest in Tesla?

2. If you add the roots of the polynomial $Px^2 + Qx + R$ together, you get a number M, and if you multiply the roots together, you get a number N. Find a polynomial with coefficients in terms of P, Q & R that has roots which add together to give N and multiply together to give M.

PROBLEM SOLVING:

- The sum of a group of n positive integers is 20.

  What is the maximum product of the group of n positive integers?

  In other words:

  $$a_1 + a_2 + a_3 + \ldots + a_n = 20$$

  Maximize the product of

  $$(a_1)(a_2)(a_3)\ldots(a_n)$$

2 Arithmetic 1

In 2000's so far, 2000-2018, the mean rainfall for March in Sacramento is 2.76 inches. How many inches of rain would Sacramento have to get in March of 2019 to raise the overall mean for the 2000's to 3 inches?

This is a classic average question, which should be clear by the use of the word "mean." As always with these kinds of questions, use the formula for the average.

> **Lemma** – The average of n numbers is given by
> $$\frac{x_1 + x_2 + \dots x_n}{n}.$$

Thus, our average for the years 2000 to 2019 is given by

$$\frac{x_{2000} + x_{2001} + \dots x_{2019}}{20}.$$

For posterity, there are *20* "March"s between (and including) 2000 and 2019 and *19* between 2000 and 2018. Count directly or note that for two integers x to y, the total number of integers and including x to y is given by $y - x + 1$.

We want to find what x_{2019} is such that

$$\frac{x_{2000} + x_{2001} + \dots x_{2019}}{20} = 3 \text{ inches.}$$

But in order to find what x_{2019} is, we need to know what $x_{2000} + x_{2001} + \dots x_{2018}$ is. Luckily, there is some information: the mean rainfall was 2.76 inches from years 2000 to 2018.

> With questions with multiple averages, you will often use that the sum of all the entries for a *particular* average is equal to the given average multiplied by the number of entries in the given average.

For example, note that the average rainfall from years 200 to 2018 can be rewritten as

$$\frac{x_{2000} + x_{2001} + \dots + x_{2018}}{19} = 2.76 \text{ inches.}$$

Multiplying both sides by 19 gives that

$$x_{2000} + x_{2001} + \dots + x_{2018} = 19 \times 2.76 \text{ inches}$$
$$\boxed{x_{2000} + x_{2001} + \dots + x_{2018} = 52.44 \text{ inches.}}$$

Then, substituting this into the average from 2000 to 2019 and equating to 3, we have that

$$\frac{52.44 \text{ inches} + x_{2019}}{20} = 3 \text{ inches.}$$

Solving, $\boxed{x_{2019} = 7.56 \text{ inches.}}$

3 Arithmetic 2

Bob went to a hamburger restaurant on a night where burgers were 50% off. He ordered a burger and sides. (The sides were priced at regular prices.) He paid 5% tax on his meal, which amounted to 62 cents. He decided to tip the server 15% of the amount he would have paid for the meal (excluding tax) if burgers were at full price. His tip was \$2.43. What is the regular price of a burger?

Note: No rounding was necessary in computing either the tax or the tip.

This is a classic "word salad" question, which entails converting all the information into equation form, then solving. The clues aren't in the easiest order – Megha, for instance doing the *second* half of the question first.

> For most word salad questions, find the easiest equations first. You do not need to go in order of the question.

First, we declare two variables: B for the full price of the burger in dollars, and S for the price of the side in dollars.

Going in Megha's order,

> *He ordered a burger and sides... He decided to tip the server* 15% *of the amount he would have paid for the meal (excluding tax) if burgers were at full price. His tip was* \$2.43.

Note that the meal, if burgers were at full price, would be the combined cost of the burger and the sides, or $B + S$. Then, a 15% tip of this full price is given by

$$0.15\,(B+S) = \$2.43.$$

Then,

> *... burgers were 50% off. He ordered a burger and sides... He paid* 5% *tax on his meal, which amounted to 62 cents.*

Note that as burgers were 50% off, the tax on his meal would be the sum of *half* the price of the burger and the price of the sides, which would be $0.5B + S$. Then, a 5% tax of this is given by

$$0.05\left(\frac{B}{2} + S\right) = \$0.62.$$

Expanding, these two equations given the following system:

$$\begin{aligned} 0.15B + 0.15S &= \$2.43 \\ 0.025B + 0.05S &= \$0.62. \end{aligned}$$

You can solve this with your preferred method, but as we only want the value of B and don't really care about S, eliminating S by multiplying the second equation by $\frac{0.15}{0.05} = 3$ is efficient.

Solving the system, the answer is $\boxed{\$7.60.}$

4 Trig/analysis 1

Define

$$T_n = 1^7 + 2^7 + 3^7 + \ldots + n^7$$

for all natural numbers n. Write the following expression in simplest form.

$$\log_{\sqrt{7}}(T_{343} - T_{342})$$

The primary difficulty in this problem is simplifying $T_{343} - T_{342}$. The key idea is to note that *since both of them share a lot of terms, subtracting them should cut a lot of them away.* It's clear if you write them out as something like

$$\begin{aligned} T_{343} &= 1^7 + 2^7 + \ldots 341^7 + 342^7 + 343^7 \\ T_{342} &= 1^7 + 2^7 + \ldots 341^7 + 342^7. \end{aligned}$$

The difference is then fairly simple to calculate:

$$\begin{aligned} T_{343} - T_{342} = \quad & 1^7 + \ldots + 342^7 + 343^7 \\ & -1^7 - \ldots - 342^7 \end{aligned}$$

Which gives that $\boxed{T_{343} - T_{342} = 343^7.}$

The rest of the problem is repeated application of logarithm rules and can be done in a multitude of ways. Substituting,

$$\log_{\sqrt{7}}(T_{343} - T_{342}) = \log_{\sqrt{7}}\left(343^7\right).$$

"Factoring" out the exponent,

$$\log_{\sqrt{7}}(T_{343} - T_{342}) = 7 \times \log_{\sqrt{7}}(343).$$

Applying the change of base formula and using the convenient base of 7,

$$\log_{\sqrt{7}}(T_{343} - T_{342}) = 7 \times \frac{\log_7 343}{\log_7 \sqrt{7}}.$$

Lemma –

$$343 = 7^3.$$

Substituting,

$$\log_{\sqrt{7}}(T_{343} - T_{342}) = 7 \times \frac{\log_7 7^3}{\log_7 7^{\frac{1}{2}}}.$$

Then using the definition of logarithms and simplifying,

$$\log_{\sqrt{7}}(T_{343} - T_{342}) = 7 \times \frac{3}{\left(\frac{1}{2}\right)}$$

$$\boxed{\log_{\sqrt{7}}(T_{343} - T_{342}) = 42.}$$

5 Trig/analysis 2

Three different math classes at a local high school each send three students on a trip to see a performance of Pi – The Musical. They are randomly assigned seats in the front row, which conveniently has 9 chairs. What is the probability that students from each class will sit together in a block of 3 consecutive seats?

First, the easy part: finding the total number of unique combinations of ways to sit 9 students in a row. $\boxed{\text{This is given by } 9!.}$

The hard part is then finding the number of unique combinations that fit requirements. Carmen did this by examining probabilities *per seat* then multiplying to make things work. This is incredibly difficult and not the *cool* way to do it. The better way is to examine *per group.*

> For questions that ask to compute number of ways to arrange something where certain elements need to be next to each other, *put those elements into groups first.*

Note that there are three classes of three people in a row of nine seats, there is only one group of ways to have the students sit in three ways together: one "block" of people from a class, another block, then another block. For concreteness, consider the row as

$$\begin{bmatrix} B_1 & B_2 & B_3 \end{bmatrix}.$$

Note that there are $\boxed{3! = 6 \text{ ways}}$ to arrange three blocks in a row.

Then, *per block*, there are three people. Concreteley, this can be represented as

$$B_n = \begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix}.$$

There are $\boxed{3! = 6 \text{ ways}}$ to arrange the three people *within the block.*

As the placements of the blocks and arrangements of students within the blocks are independent, there are $\boxed{6 \times 6 \times 6 \times 6 = 6^4 \text{ arrangements}}$ which satisfy the requirements.

Dividing this by the total number of unique arrangements gives the desired answer of $\boxed{\frac{6^4}{9!}.}$

6 Geometry 1

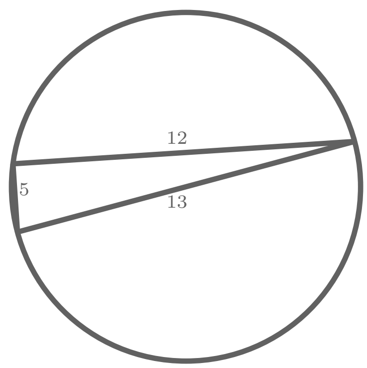
A triangle with sides 5, 12, and 13 is inscribed in a circle. What is the circumference of the circle?

Lemma – A 5, 12, 13 triangle is a Pythagorean triple and thus a right triangle.

This was the very first thing I noticed.

Lemma – If a right triangle is inscribed in a circle, then the hypotenuse of the triangle is a diameter.

I actually assumed that this is true for all triangles, but this is not correct. Luckily the triangle is a right triangle and this holds. Though not necessary, drawing this gives



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As the diameter is 13, multiplying by π gives the circumference of $\boxed{13\pi.}$

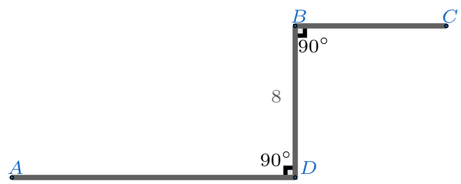
7 Geometry 2

Given: Quadrilateral $ABCD$. $\overline{BD} \perp \overline{AD}$, $\overline{BD} \perp \overline{BC}$, $BD = 8$, $AB = 17$, $CD = 10$. What is the area of Quadrilateral $ABCD$?

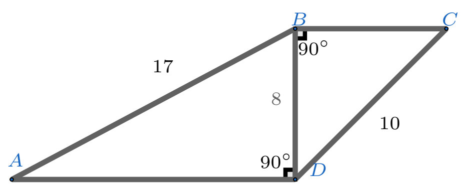
This question is somewhat difficult solely because drawing the diagram accurately takes a bit of work. Drawing a rectangle gets the perpendicular parts correctly, but is then *way* off in getting the lengths right. This is a good learning opportunity for drawing things in a convenient order.

> In general, do not draw things in order. Try to draw things with the most restrictions first.

Note that $\overline{BD}$ is the most "involved" segment, as it is both perpendicular to $\overline{AD}$ and $\overline{BC}$. Drawing $\overline{BD}$ first, then drawing $\overline{AD}$ and $\overline{BC}$ should give you something like the following:



I cheated as I already know the lengths, having done the problem, but the principle still applies. Note further that point A and C must be going in the "opposite" direction from $\overline{BD}$, as since it's Quadrilateral $ABCD$, the point must run in that order either clockwise or counterclockwise. Connecting points to "finish" Quadrilateral $ABCD$ and marking lengths "updates" the diagram giving



Note that the quadrilateral is split into two right triangles, both having height 8, so what's left is to find the bases, $\overline{AD}$ and $\overline{BC}$. Alternatively *don't* notice this, and just blindly solve for more sides and realize this halfway. As both triangle $\triangle ABD$ and $\triangle BCD$ are right triangles, we proceed with the Pythagorean theorem.

$$\overline{AB}^2 = \overline{AD}^2 + \overline{CD}^2$$
$$17^2 = \overline{AD}^2 + 8^2$$
$$289 = \overline{AD}^2 + 64$$
$$225 = \overline{AD}^2$$
$$\boxed{\overline{AD} = 15.}$$

Likewise for $\overline{BC}$,

$$\begin{aligned}
\overline{CD}^2 &= \overline{BC}^2 + \overline{BD}^2 \\
10^2 &= \overline{BC}^2 + 8^2 \\
100 &= \overline{BC}^2 + 64 \\
36 &= \overline{BC}^2 \\
&\boxed{\overline{BC} = 6.}
\end{aligned}$$

This is finished by using the hipster method of the trapezoid area formula or simply summing the area of the two triangles. Summing,

$$\begin{aligned}
&ABCD = \triangle ABD + \triangle BCD \\
&ABCD = \frac{1}{2}(15)(8) + \frac{1}{2}(6)(8) \\
&ABCD = 60 + 24 \\
&\boxed{ABCD = 84.}
\end{aligned}$$

8 Algebra 1

Olen Muks invests a certain amount dollars in Tesla, Inc stock. To his delight, his investment increased by 10%. Being the cautious investor, he then takes \$130 of his investment out of the stock market and hides it under his mattress. The amount of his investment remaining in Tesla stock then decreases by 10%. Olen then pulls all of his money out of the stock and puts it under his mattress. The amount of money under his mattress is 3% more than what he originally invested. How much did Olen Muks originally invest in Tesla?

This is another classical word salad question.

> For questions where some initial quantity is being repeatedly change, proceed in order.

Proceeding in order,

> *Olen Muks invests a certain amount dollars in Tesla, Inc stock.*

Call this initial quantity x, and the current amount of money A. As of now, his current amount of money is exactly what he put in, or

$$A = x.$$

> *...his investment increased by* 10%.

The initial quantity x has increased by 10%, so the current quantity is now

$$A = 1.1x.$$

> *...he then takes* \$130 *of his investment out of the stock market and hides it under his mattress.*

The quantity has now decreased by \$130, so the current quantity is now

$$A = 1.1x - \$130.$$

> *...his investment remaining in Tesla stock then decreases by* 10%.

The entire current quantity decreases by 10%, so the current amount of money is now

$$A = 0.9\left(1.1x - \$130\right).$$

> *...The amount of money under his mattress is* 3% *more than what he originally invested.*

This notes that the current amount of money A, is $1.03x$. Substituting the previous expression for A and putting it all together gives

$$0.9\left(1.1x - \$130\right) = 1.03x.$$

Expanding and solving gives that the initial quantity invested is $\boxed{\$325.}$

9 Algebra 2

If you add the roots of the polynomial $Px^2 + Qx + R$ together, you get a number M, and if you multiply the roots together, you get a number N. Find a polynomial with coefficients in terms of P, Q & R that has roots which add together to give N and multiply together to give M.

This question is *very* trivial if you know Vieta's formulas, but quite challenging if you do not. Carmen tried to bash this with the quadratic formula (which actually *is* doable with some tricks), but is still nowhere near as easy as with Vieta.

If you ever see a question mention the word sum and/or product of the roots of a polynomial, attempt to use Vieta immediately.

Shadid saw the question and quipped that the problem writers were clearly running out of ideas.

Lemma – For a given polynomial of degree *at least* 1

$$P(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0,$$

the product of the roots is given by a_0, and the sum is given by $-a_1$.

Note that we cannot readily apply this form of Vieta's formula to the problem, as the leading term of the polynomial $Px^2 + Qx + R$ is not 1. To fix this, we divide all terms by P, giving the equivalent polynomial

$$x^2 + \frac{Q}{P}x + \frac{R}{P}.$$

Applying Vieta's formulas here, the sum of the roots is given by

$$M = -\frac{Q}{P}$$

and the product of all the roots is given by

$$N = \frac{R}{P}.$$

And for the "new" polynomial, which we can say to be

$$x^2 + bx + c,$$

would have roots summing to N and multiplying to M. Applying Vieta's formula, we have that $c = M$ and $-b = N$.

Substituting, $c = -\frac{Q}{P}$ and $-b = \frac{R}{P}$. Solving for b by multiplying both sides by -1 then substituting again, we have that the new polynomial is

$$\boxed{x^2 - \frac{R}{P}x - \frac{Q}{P}.}$$

10 Problem solving

The sum of a group of n positive integers is 20.

What is the maximum product of the group of n positive integers?

In other words:

$$a_1 + a_2 + a_3 + \ldots + a_n = 20$$

Maximize the product of

$$(a_1)(a_2)(a_3)\ldots(a_n)$$

> For problems where you have the option of picking different numbers among a fixed number of numbers which are then evaluated on some function, you will either want to the numbers as close as possible, or as many large numbers as possible.

The rigorous version of this statement exists within Jensen's inequality I believe, but I am not sure. For multiplication however, you will want to have numbers as close as possible over having a few really large numbers and a few really small numbers.

Here's a small demonstration (not a proof):

$$3 \times 3 > 1 \times 5.$$

Here each pair of numbers is restricted in that the two numbers could not exceed 6. The pair in which the two numbers were closer ended up greater than the pair in which one large number existed.

With this in mind, you really only need to check possibilities until you see a decrease, which is not too difficult (I myself was able to do it alone.) There's a useful heuristic, pointed out to me by Wilson:

> **Lemma –** The maximal product for n is approximated by
>
> $$\left(\frac{20}{n}\right)^n.$$

The idea for this approximation is that you if try to keep the numbers as close as possible, for a given n, you'll be multiplying $\frac{20}{n}$ by itself, n times.

Just looking this up, this is maximized for $n = e$ by logarithmic differentiation, which means approximately $\frac{20}{e} \approx 7.36$ numbers is optimal. This suggests either $n = 7$ or $n = 8$.

However that's not at all what I did. I just checked until I saw a decrease.

> On group questions, if you have a single hunch that can be "solved" by simply checking not too large number of cases, it can often be more efficient to simply have everyone checking than trying to generate some whimsical insight.

The methodology is to divide 20 by n, but seems problematic as sometimes you end up with a fraction. For instance, with $n = 3$, this gives $\frac{20}{3} \approx 6.6$. This can be worked around by simply rounding down, giving

$$\begin{bmatrix} 6 & 6 & 6 \end{bmatrix}$$

as an intermediate step. However, these three numbers do not sum to 20. To remedy this, increment numbers by 1 until they do. We increment by 1 in order to keep the numbers as close as possible. After this, the set becomes

$$\begin{bmatrix} 6 & 7 & 7 \end{bmatrix},$$

which we then compute the product to be 294.

Repeating until a decrease,

n	Numbers	Product
1	20	20
2	10, 10	100
3	6, 7, 7	294
4	5, 5, 5, 5	625
5	4, 4, 4, 4, 4	1024
6	3, 3, 3, 3, 4, 4	1296
7	2, 3, 3, 3, 3, 3, 3	1458
8	2, 2, 2, 2, 3, 3, 3, 3	1296

As 1458 is the largest product, the maximum product is $\boxed{1458.}$

Technically speaking, this methodology is sufficient if and only if the function is concave down. To be completely rigorous, simply compute the other 12 cases.