Integration by Parts

DeVon Herr [*]

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Abstract

In this article we derive the famous "integration by parts" formula $\int udv = uv - \int vdu$, give intuition on said derivation, then apply the formula to solve problems, including . We also discuss some mnemonics for the choice of u and dv, tabular integration or "tic-tac-toe" integration of *Stand and Deliver* fame as well as "infinite" problems.

Dedicated to Erica and Matthew

1 Introduction

By the fundamental theorem of calculus, the process of anti-differentiation and integration are intimately linked: for a continuous function on a given interval, the integral is the anti-derivative. As the word anti-derivative implies, finding the integral is then simply finding another function whose derivative is the original function. For this specific reason, most "rules" and "formulas" for integration are really just differentiation rules backwards.

The most explicit example is the power rule. For any function x^n, the derivative is given by

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Verbally, the derivative is computed by multiplying the function by the exponent, then subtracting one from the exponent. Likewise, the integral of a function x^n is the process backwards.

$$\int x^n \; dx = \frac{1}{n+1}x^{n+1} + C.$$

One now adds one to the exponent, then divides by the new exponent.

Integration by substitution (or $u-$sub) is the integration counterpart of the chain-rule. We seek, then, to find the counterpart for the product-rule

$$\frac{d}{dx}f(x) \times g(x) = (f'(x) \times g(x)) + (f(x) \times g'(x)).$$

[*]Special thanks to Mr. Steele, who taught me this first, Lexi, for strengthening my resolve, and Erica, who gave my words a purpose.

2 A Motivating Example

Example 1

Find the integral of

$$f'(x) = x^3 e^x + 3x^2 e^x,$$

with respect to x.

A pretty big clue is that the anti-derivative probably involves the functions x^3 and e^x; note that the derivative of x^3 is $3x^2$ and the derivative of e^x is, well, e^x.

A natural guess, then, is that the integral of the function is $f(x) = x^3 e^x$. Indeed, taking the derivative of $x^3 e^x$ verifies that

$$\frac{d}{dx}\left(x^3 e^x\right) = x^3 e^x + 3x^2 e^x.$$

For harder examples, the thought process of "what functions are involved?" will remain useful. In the spirit of raising the difficulty of the problem but keeping details the same, find the anti-derivative of just *one* of the terms.

Example 2

Find the anti-derivative of

$$g'(x) = 3x^2 e^x.$$

While the product rule of derivatives yields a sum of two terms, unfortunately, this is only a single term. For integrals that are a product of two distinct functions, we require a new trick.

2.1 First Steps towards Derivation

Writing the explicit differentiation again, but this time with colors,

$$\frac{d}{dx}\left(x^3 e^x\right) = {\color{blue}x^3 e^x} + {\color{red}3x^2 e^x.}$$

Integrating both sides of the expression,

$$\int \left(\frac{d}{dx}\left(x^3 e^x\right)\right)\, dx = \int {\color{blue}x^3 e^x} + {\color{red}3x^2 e^x}\, dx.$$

Noting that the anti-derivative of a derivative is just the original function itself,

$$x^3 e^x = \int {\color{blue}x^3 e^x} + {\color{red}3x^2 e^x}\, dx.$$

We're getting closer. The last leap of faith is to note that the integral of the sum of two functions can be split into two integrals. Doing so,

$$x^3 e^x = \int x^3 e^x \, dx + \int 3x^2 e^x \, dx.$$

Aha! Look again. **The integral of $x^3 e^x$ is precisely what we're after.** Algebraically solving for $\int x^3 e^x \, dx$,

$$\int x^3 e^x \, dx = x^3 e^x - \int 3x^2 e^x \, dx.$$

Unfortunately, we're stuck, again. We don't have a way to actually simplify $\int 3x^2 e^x \, dx$ yet. Keep this algebraic manipulation in mind, however. We will use this again later.

3 Deriving the Integration by Parts Formula

Consider two functions $u(x)$ and $v(x)$, or u and v for short. Their derivatives will thus be denoted by du and dv. Taking the derivative of $u(x) \times v(x)$ by the product rule,

$$\frac{d}{dx} u(x)v(x) = u'(x)v(x) + u(x)v'(x).$$

Following the same procedure the subsection prior, we integrate both sides.

$$u(x)v(x) = \int u'(x)v(x) \, dx + \int u(x)v'(x) \, dx.$$

Solving for one of the terms, say $\int u(x)v'(x) \, dx$, gives

$$\int u(x)v'(x) \, dx = u(x)v(x) - \int u'(x)v(x).$$

Rearranging and using short-hands, we have the integration by parts formula:

$$\boxed{\int udv = uv - \int vdu.}$$

4 Simple Worked Examples

Try to follow along with each step.

Example 3

Evaluate

$$\int xe^x \, dx.$$

Note that the only reason this integral is worth looking at is that it's a product of two functions that don't talk to each other. If the function was just e^x, it would be easy. If it was just x, it would be easy. Even if it were $2xe^{x^2}$, it would be easy, as x^2 and $2x$ *do* talk to each other, $2x$ is x^2's derivative.

Recalling the integration by parts formula $\int udv = uv - \int vdu$, we will have to make a choice before we can began integrating. If $\int xe^x \, dx$ is the $\int udv$ term, we will have to pick one function to be u and one to be dv.

In this example, we will let $u = x$ and $dv = e^x dx$. Don't worry about which to choose now, in this example we will focus more on the actual mechanics of integration by parts.

$$\begin{aligned} u &= x \\ dv &= e^x dx. \end{aligned}$$

Taking yet another look at the integration by parts formula, in addition to u and dv, the expression demands du and v. We only have u and dv so far, so we'll have to find what du and v are from them. Remember in when we derived the integration by parts formula last section, we had that du is the derivative of u and dv is the integral of v. Taking the derivative of u,

$$\begin{aligned} u &= x, \qquad du = 1 \; dx \\ dv &= e^x \; dx. \end{aligned}$$

Now to find v. Note that if dv is the derivative of v, this is the same thing as saying v is the integral of dv. Integrating dv,

$$\begin{aligned} u \; &= x, \qquad & du &= 1 \; dx \\ dv &= e^x \; dx \qquad & v \; &= e^x. \end{aligned}$$

Armed with everything, we can now use the integration by parts formula.

$$\begin{aligned} \int udv &= uv - \int vdu \\ \int xe^x \; dx &= x \times e^x - \int e^x \times 1 \; dx. \end{aligned}$$

Simplifying,

$$\int xe^x \; dx = xe^x - \int e^x \; dx.$$

All that's left to do is to compute the integral of e^x, which we know to be $e^x + C$. Integrating,

$$\int xe^x \; dx = xe^x - e^x + C.$$

And we are done.

Let's try a harder one.

Example 4

Evaluate
$$\int x\sqrt{x+1}\ dx.$$
Hint: Let $u = x$ and $dv = \sqrt{x+1}\ dx$.

We start by differentiating u to find du and integrating dv to find v.

$$\begin{aligned} u &= x & du &= 1\ dx \\ dv &= \sqrt{x+1}\ dx & v &= \frac{2}{3}(x+1)^{\frac{3}{2}} \end{aligned}$$

Applying the formula,

$$\int udv = uv - \int vdu$$
$$\int x\sqrt{x+1}\ dx = x \times \left(\frac{2}{3}(x+1)^{\frac{3}{2}}\right) - \int \frac{2}{3}(x+1)^{\frac{3}{2}} \times 1\ dx.$$

Simplifying,

$$\int x\sqrt{x+1}\ dx = \frac{2x}{3}(x+1)^{\frac{3}{2}} - \int \frac{2}{3}(x+1)^{\frac{3}{2}}\ dx.$$

The last step is to evaluate the integral of $\frac{2}{3}(x+1)^{\frac{3}{2}}$.

$$\int x\sqrt{x+1}\ dx = \frac{2x}{3}(x+1)^{\frac{3}{2}} - \frac{4}{15}(x+1)^{\frac{5}{2}} + C.$$

And we are done.

5 Picking u and dv

I have held off from talking about selecting functions to be u and dv as it should come secondary to the actual differentiation and integration processes necessary for integration by parts.

Reviewing the integration by parts formula

$$\int udv = uv - \int vdu,$$

finding the uv term by integrating dv to find v is generally easy, but finding the $-\int vdu$ term can be considerably difficult.

Why? Integrating a single function is generally straight forward, and then multiplying it with another function is not too bad either. However, multiplying two functions together *then* integrating can be difficult if the product is particularly "bad." In fact, integrating products of functions is what integration by parts is actually used for, so sometimes integration by parts has to be used *again* just to simplify the $-\int v du$ part.

We want to make our lives easier, and make sure the $-\int v du$ term is as simple as possible. To do so, we follow a mnemonic:

"L.I.P.E.T."

Another one is

"L.I.A.T.E."

Both of them use the same principle.

L.I.P.E.T.

- L – Logarithmic
- I – Inverse trigonometric
- P – Polynomial
- E – Exponential
- T – Trigonometric

The idea is that when given the product of two functions, one should classify the two functions into one of the 5 types, and whichever comes first in the word "LIPET" will be u, and the other function should be dv.

Example 5

What should u and dv be in order to integrate

$$f(x) = x \ln(x)$$

with respect to x?

The two functions involved here are x and $\ln(x)$. The former is a polynomial, and the latter is a logarithmic function. As L comes before P in "LIPET," $u = \ln(x)$ and $dv = x$.

It may be worthwhile to try and actually integrate $x \ln(x)$ for practice.

Example 6

What should u and dv be in order to integrate

$$\int x\cos(x)\ dx$$

with respect to x?

The two functions are x and $\cos(x)$. The former is a polynomial, and the latter is a trigonometric function. As P comes before T in "LIPET," $u = x$ and $dv = \cos(x)$.

It may be worthwhile to try and actually integrate $x\cos(x)$, too, for practice.

6 Definite Integration

Luckily for us, unlike certain $u-$sub questions, the upper and lower limits of integration don't do anything weird with integration by parts. That is, if one takes the definite integral of a product of two functions $u(x) \times v'(x)$,

$$\int_a^b u(x) \times v'(x) = u(x) \times v(x)|_a^b - \int_a^b v(x) \times u'(x)\ dx,$$

or more compactly,

$$\int_a^b udv = uv|_a^b - \int_a^b vdu.$$

That is, one can find the anti-derivative of a product of functions by integration by parts, and then apply the upper and lower limits of integration to the anti-derivative.

7 Iterated Integration by Parts

We discussed earlier that sometimes the $-\int vdu$ term doesn't integrate easily and may actually require another integration by parts. While this doesn't require any new tricks or new math, it is good to be prepared for it and be exposed.

Example 7

Evaluate the following integral.

$$\int w^2\sin(10w)\ dw$$

By "LIPET," $u = w^2$ and $dv = \sin(10w)\ dw$. Differentiating u to find du and integrating dv to find v gives us

$$\begin{aligned} u &= w^2 & du &= 2w\ dw \\ dv &= \sin(10w)\ dw & v &= \frac{-1}{10}\cos(10w). \end{aligned}$$

Using the integration by parts formula,

$$\int u dv = uv - \int v du$$

$$\int w^2 \sin(10w)\ dw = w^2 \times \frac{-1}{10}\cos(10w) - \int \frac{-1}{10}\cos(10w) \times 2w\ dw.$$

Simplifying terms,

$$\int w^2 \sin(10w)\ dw = \frac{-w^2}{10}\cos(10w) + \int \frac{w}{5}\cos(10w)\ dw.$$

Note that factoring out the -1 from the $\frac{-1}{10}$ turns the integral positive.

We now hit a roadblock, as $\int \frac{w}{5}\cos(10w)\ dw$ doesn't easily integrate. In fact, as a product of two functions that don't talk to each other, **we actually have to do integration by parts again.**

Noting that $\frac{w}{5}$ is a polynomial (a linear one) and $\cos(10w)$ is a trigonometric function, by "LIPET," $u = \frac{w}{5}$ and $dv = \cos(10w)\ dw$. Differentiating u to find du and v to find dv gives us

$$\begin{aligned} u &= \frac{w}{5} & du &= \frac{1}{5}\ dw \\ dv &= \cos(10w) & v &= \frac{1}{10}\sin(10w). \end{aligned}$$

Using the integration by parts formula again,

$$\int u dv = uv - \int v du$$

$$\int \frac{w}{5}\cos(10w)\ dw = \frac{w}{5} \times \frac{1}{10}\sin(10w) - \int \frac{1}{10}\sin(10w) \times \frac{1}{5}\ dw.$$

Simplifying terms,

$$\int \frac{w}{5}\cos(10w)\ dw = \frac{w}{50}\sin(10w) - \int \frac{1}{50}\sin(10w)\ dw.$$

All that's left is to carry out integrating $\frac{1}{50}\sin(10w)$.

$$\int \frac{w}{5}\cos(10w)\ dw = \frac{w}{50}\sin(10w) + \frac{1}{500}\cos(10w) + C.$$

We're not done yet. We've now figured out what $\int \frac{w}{5}\cos(10w)\ dw$ is, but remember, this was just a sub-problem to find the integral of $\int w^2 \sin(10w)\ dw$, which we found to be equal to $\frac{-w^2}{10}\cos(10w) + \int \frac{w}{5}\cos(10w)\ dw$. Putting everything together,

$$\int w^2 \sin(10w)\ dw = \frac{-w^2}{10}\cos(10w) + \frac{w}{50}\sin(10w) + \frac{1}{500}\cos(10w) + C.$$

And we are done.

7.1 Tabular Integration

An often easier way to deal with iterated integration by parts is a method known as tabular integration, for calculations use a table or "tic-tac-toe" of *Stand and Deliver* fame, due to three column nature of the calculations. Instead of constantly generating new integrals, this method brainlessly generates higher order derivatives and integrals of the original functions and avoids the potential mess that comes from constant substitution.

Example 8

Evaluate the following integral.

$$\int x^4 e^{\frac{x}{2}}\ dx$$

A quick "LIPET" test tells us that $u = x^4$ and $dv = e^{\frac{x}{2}}$. To proceed, we construct a table as follows.

(Sign)	u	dv
$+$	x^4	$e^{\frac{x}{2}}$.

Then, just like in a normal integration by parts problem, we take the derivative of u and the integral of dv. However, we then add them to the row underneath the current one. For the sign column, we change it.

(Sign)	u	dv
$+$	x^4	$e^{\frac{x}{2}}$
$-$	$4x^3$	$2e^{\frac{x}{2}}$.

We keep repeating until the u column reaches 0.

(Sign)	u	dv
$+$	x^4	$e^{\frac{x}{2}}$
$-$	$4x^3$	$2e^{\frac{x}{2}}$
$+$	$12x^2$	$4e^{\frac{x}{2}}$
$-$	$24x$	$8e^{\frac{x}{2}}$
$+$	24	$16e^{\frac{x}{2}}$
$-$	0	$32e^{\frac{x}{2}}$

Once the table has been constructed, we do a series of multiplications. Starting with the sign column, the entry is multiplied by the u column entry one to the right, and then by the dv column one to right *and down*. The first term from this process would be $(+) \times x^4 \times 2e^{\frac{x}{2}} = 2x^4 e^{\frac{x}{2}}$, and the next would be $(-) \times 4x^3 \times 4e^{\frac{x}{2}} = -16x^3 e^{\frac{x}{2}}$.

Graphically, one follows the lines below, going one to the right, before curving downwards, multiplying as one goes.

(Sign)	u	dv	Result
$+$	x^4	$e^{\frac{x}{2}}$	
$-$	$4x^3$	$2e^{\frac{x}{2}}$	$2x^4 e^{\frac{x}{2}}$
$+$	$12x^2$	$4e^{\frac{x}{2}}$	$-16x^3 e^{\frac{x}{2}}$
$-$	$24x$	$8e^{\frac{x}{2}}$	$96e^{\frac{x}{2}}$
$+$	24	$16e^{\frac{x}{2}}$	$-384e^{\frac{x}{2}}$
$-$	0	$32e^{\frac{x}{2}}$	$768e^{\frac{x}{2}}$

Then one sums up all the resultant terms, and then adds the arbitrary constant of integration. Thus,

$$\int x^4 e^{\frac{x}{2}}\ dx = 2x^4 e^{\frac{x}{2}} - 16x^3 e^{\frac{x}{2}} + 96x^2 e^{\frac{x}{2}} - 384x e^{\frac{x}{2}} + 768 e^{\frac{x}{2}} + C.$$

And we are done.

Note that this technique can also replace standard integration by parts procedure, and may even be faster for some as well.

7.2 Cyclic Integrals

When trying to integrate functions that are combinations of trigonometric or exponential functions, the $-\int vdu$ term in the integration by parts will need another integral which will need yet another integral and so on.

Example 9

Find the anti-derivative of

$$f'(\theta) = e^{\theta} \cos(\theta),$$

with respect to θ.

Things get a little crazy here. Standard integration by parts and tabular integration actually both fail, as e^θ and $\cos(\theta)$ can be differentiated and integrated forever without ever hitting 0. For instance, using $u = e^\theta$ and $dv = \cos(\theta)\ d\theta$,

(Sign)	u	dv
+	e^θ	$\cos(\theta)$
-	e^θ	$\sin(\theta)$
+	e^θ	$-\cos(\theta)$
-	e^θ	$-\sin(\theta)$
+	e^θ	$\cos(\theta)$
$\vdots$	$\vdots$	$\vdots$

Neither term, even if one switches which function is u and which is dv, ever reaches 0. We need a new trick.

Consider integrating by parts explicitly:

$$\int e^\theta \cos\theta\ d\theta = e^\theta \sin(\theta) - \int \sin(\theta) e^\theta\ d\theta.$$

Integrating $\sin(\theta)e^\theta$ by parts,

$$\int e^\theta \cos\theta\ d\theta = e^\theta \sin(\theta) - \left(-e^\theta \cos(\theta) - \int -e^\theta \cos(\theta)\ d\theta\right).$$

Simplifying quite a few negatives,

$$\int e^\theta \cos\theta\ d\theta = e^\theta \sin(\theta) + e^\theta \cos(\theta) - \int e^\theta \cos(\theta)\ d\theta.$$

Hey! The desired expression, $\int e^\theta \cos(\theta)\ d\theta$, is on both sides! Just like how we could solve something like

$$x = 3 - x$$

by adding x, the desired quantity to both sides, then solve for it. Following the same logic, we will add $\int e^\theta \cos(\theta)\ d\theta$ to both sides. Adding,

$$2\int e^\theta \cos(\theta)\ d\theta = e^\theta \cos(\theta) + e^\theta \sin(\theta) + C.$$

For unimportant reasons, we add the constant of integration in the prior step. Then, to isolate the desired expression, one then divides by 2.

$$\int e^\theta \cos(\theta)\ d\theta = \frac{e^\theta \cos(\theta) + e^\theta \sin(\theta)}{2} + C.$$

And we are done.

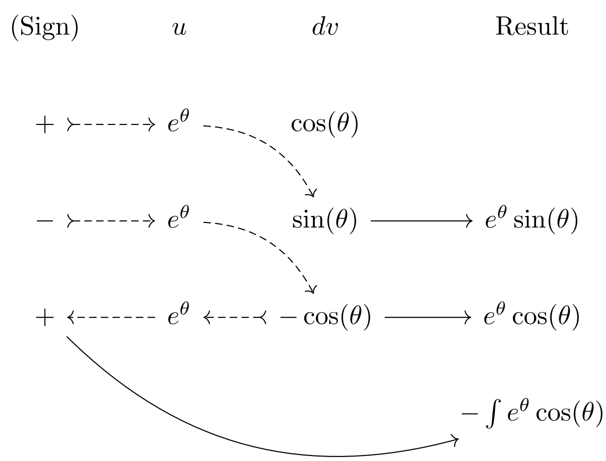
By modifying the procedure for tabular integration, tabular integration can also be used. Instead of repeatedly differentiating and integrating until a column hits 0, we differentiate and integrate until we see the original function again. This function can be either negative or positive, or a multiple of the original function, so long as it contains the original function.

In this example, it would be

(Sign)	u	dv
+	e^{θ}	$\cos(\theta)$
-	e^{θ}	$\sin(\theta)$
+	e^{θ}	$-\cos(\theta)$

Here, we stop as we see the $\cos(\theta)$ function appear again in the dv column. Again, it is not a problem if the function is negative.

The multiplication order runs a bit differently. In the very last column, where one sees a (potentially modified) copy of an original function, one writes the integral of the product of that function, and all terms to the left.



Graphically, one starts from the left column, and follows the path of arrows from the tail to the head. The copy, however, has a special path going left, which is then integrated. The sum of these terms, then, is equated to the original integral, giving

$$\int e^{\theta}\cos\theta \; d\theta = e^{\theta}\sin(\theta) + e^{\theta}\cos(\theta) - \int e^{\theta}\cos(\theta) \; d\theta.$$

Adding $\int e^{\theta}\cos(\theta) \; d\theta$ to both sides gives

$$2\int e^{\theta}\cos\theta \; d\theta = e^{\theta}\sin(\theta) + e^{\theta}\cos(\theta) + C,$$

which simplifies to

$$\int e^{\theta}\cos\theta \; d\theta = \frac{e^{\theta}\sin(\theta) + e^{\theta}\cos(\theta)}{2} + C.$$

And we are done.

8 Weird, Single Functions

Despite integration by parts usually anti-differentiating products of two functions, there are times where integration by parts is necessary to integrate a single function.

Example 10

Evaluate the following integral.

$$\int \ln(x)\ dx$$

(It is *not* $\frac{1}{x}$! That is the derivative of $\ln(x)$!)

The idea with integrals such as these is to **view these as a function multiplied by 1.** In this case, it would be

$$f(x) = \ln(x) \times 1,$$

the expression which we would have to use integration by parts on.

Since 1 is technically a polynomial (of degree zero) and $\ln(x)$ is a logarithmic function, $u = \ln(x)$ and $dv = 1\ dx$. Then,

$$\begin{aligned} u &= \ln(x) \qquad & du &= \frac{1}{x}\ dx \\ dv &= 1\ dx \qquad & v &= x. \end{aligned}$$

By the integration by parts formula,

$$\int udv = uv - \int vdu$$
$$\int \ln(x) \times 1\ dx = \ln(x) \times x - \int x \times \frac{1}{x}\ dx.$$

Simplifying,

$$\int \ln(x)\ dx = x\ln(x) - \int 1\ dx.$$

Integrating 1,

$$\int \ln(x)\ dx = x\ln(x) - x + C.$$

And we are done.

9 Practice Problems

1.
$$\int 4x\cos(2-3x)\ dx$$

2.
$$\int_6^0 (2+5x)e^{\frac{1}{3}x}\ dx$$

3.
$$\int \left(3+t^2\right)\sin(2t)\ dt$$

4.
$$\int 6\tan^{-1}\left(\frac{8}{w}\right)\ dw$$

5.
$$\int e^{2z}\cos\left(\frac{1}{4}z\right)\ dz$$

6.
$$\int_0^\pi x^2\cos(4x)\ dx$$

7.
$$\int t^7\sin\left(2t^4\right)\ dt$$

8.
$$\int y^6\cos(3y)\ dy$$

9.
$$\int \left(4x^3-9x^2+7x+3\right)e^{-x}\ dx$$