Finding Absolute Extrema on a Closed Interval

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1 Introduction

In general, finding the relative minima and maxima of a function in AP calculus is fairly straightforward; set the derivative of the function to zero, and make use of the second derivative test (or by inspecting the sign changes) to filter out the more annoying cases (inflection points).

However, for more pathological expressions (i.e. *really* gross looking things), especially those where an integral is involved and calculator use is necessary, things get much bloodier. Often time a question on either AP Calculus test asks for something seemingly simple: **To find the absolute minimum/maximum value on some interval.**

So, let's get into it!

2 Machinery

Hopefully, you should be well versed with what a relative/local and absolute/global minima/maxima values are. To give short definitions:

- Relative/local minima/maxima values are where, somewhere surrounding the point, no points are less/greater than the point.
- Absolute/global minima/maxima values are where in some specified interval, it is literally the smallest/largest value.

To move forward with finding absolute extrema, we do need to unpack some machinery that makes doing so possible.

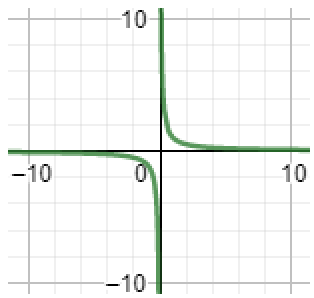
The Extreme Value Theorem

Formally: If a function $f(x)$ is continuous on a closed interval $[a, b]$, then there are two numbers $a \leq c$, $d \leq b$ in the interval such that $f(c)$ is a minimum and $f(d)$ is a maximum.

Informally: If a function is continuous on some interval, there must exist some minima and maxima on it.

This theorem needs to be qualified in two directions for our use. For one, if a function is discontinuous on some given domain, it may not have any absolute extrema. Consider the following example:

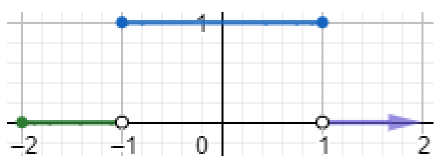
$$f(x) = \frac{1}{x} \qquad x \in [-1, 1]$$



Since the function infinitely decreases as it approaches $x = 0$ from the left and infinitely increases as it approaches $x = 1$ from the right, there is no one smallest or largest value in this interval.

However, discontinuity is not damning. Consider the following example.

$$g(x) = \begin{cases} 0 \text{ if } -2 \leq x \leq -1 \\ 1 \text{ if } -1 < x < \ \ 1 \\ 0 \text{ if } \quad 1 \leq x \end{cases} \qquad x \in [-2, 2]$$



By inspection, the function is discontinuous at $x = -1$ and $x = 1$. However, it still has an absolute maximum of 1 and an absolute minimum of 0 (it has these values at multiple x-values, however).

This just means one has to be careful. A function, if continuous on some closed interval, will necessarily have a minimal and maximal value. If not, however, it *may* have none, one or both.

Fermat's Theorem

Formally: If $f(x)$ has a relative extrema at some point $x = c$ and $f'(c)$ exists, then $x = c$ is a critical point and $f'(c)$ must equal 0.

Informally: All relative extrema are critical points.

This is useful! Since all relative extrema are critical points, **if given a list of all of a functions critical points, every relative extrema must be in that list.**

However, the converse is **not** true. Having a critical point does *not* guarantee that the point is a relative extrema. For instance, consider the following function and its first few order derivatives:

$$\begin{aligned} h(x) &= x^5 \\ h'(x) &= 5x^4 \\ h''(x) &= 20x^3 \end{aligned}$$

By algebra, $h'(0) = 0$ so $h(x)$ has a critical point at $x = 0$. $h''(0) = 0$, so by the second derivative test, it is not an extremum (and is an inflection point).

"The Endpoint Lemma" [a]

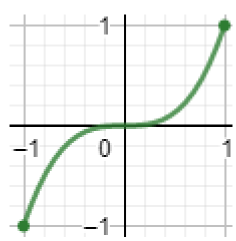
[a] There's actually not a name for this that I could find, hence the quotation marks.

Semi-formally: On a given closed interval $[a, b]$, a function $i(x)$ *may* have absolute extrema at point $x = a$ or $x = b$.

Informally: The endpoints of an interval may be extrema.

Consider the following function on a given interval.

$$j(x) = x^3 \quad x \in [-1, 1]$$



For one, the absolute extrema are precisely the endpoints $x = -1$ and $x = 1$. Additionally, the extrema do not lie on the critical points of the function, as there are none.

This just means that **when discerning the absolute extrema of a function, one also needs to check the endpoints.**

3 Laying Plans

The method is as follows.

Plan of Attack: "The Candidates Test"

1. **Acknowledge interval and discern if the function is continuous on said interval.**

2. **Generate a list of all critical points within the interval.**

3. **Plug in all critical points *and* the endpoints. These are your so-called, "candidates."**

4. **Pick the smallest/largest value.**

This is sufficient, as **if an absolute extrema exists on an interval for some function, it is either a critical point by Fermat's theorem, or an endpoint by "The endpoint lemma."** By generating a list of both the critical points and the endpoints, this covers all possibilities.

Evaluating all critical points will then show, of all possible points, which is the smallest/largest.

4 A Straightforward Example

Question 1 AP Calculus AB 2013

(CALCULATOR ACTIVE) On a certain workday, the rate, in tons per hour, at which unprocessed gravel arrives at a gravel processing plant is modeled by $G(t) = 90 + 45\cos\left(\frac{t^2}{18}\right)$, where t is measured in hours and $0 \leq t \leq 8$. At the beginning of the workday ($t = 0$), the plant has 500 tons of unprocessed gravel. During the hours of operation, $0 \leq t \leq 8$, the plant processes gravel at a constant rate of 100 tons per hour.

What is the maximum amount of unprocessed gravel at the plant during the hours of operation on this workday? Justify your answer.

The first step is to gauge the interval and see if the function is continuous on the interval. The interval is $[0, 8]$.

Finding the function is a little bit involved, but is straightforward if one is familiar with the concept of "in-out" functions. The function is the following.

$$\begin{aligned} A'(t) &= \text{in}'(t) - \text{out}'(t) \\ A'(t) &= G(t) - 100 \end{aligned}$$

$$A(x) = \int_0^x A'(t)\ dt = \int_0^x G(t) - 100\ dt + 500$$

Since this is a calculator active question, one can simply inspect the graph on a calculator. The other way is to notice that the functions are "smooth", so an algebraic combination of them will also be smooth.

The next step is to generate a list of all critical points. This is just where the derivative of the relevant function changes sign, or in this case,

$$A'(t) = G(t) - 100$$

Using a graphing calculator or CAS produces the following solution within the interval $[0, 8]$.

$$t = 4.923...$$

Including endpoints, this means our list of candidates for absolute maximum are the following.

$$t = 0, 4.923...\ , 8$$

To identify which is the maximum, one just has to evaluate the function at each of these values and take the largest.

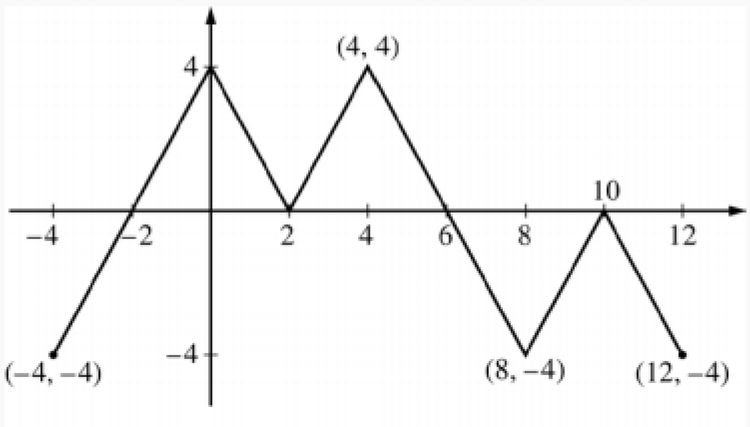
x	$A(x)$
0	$\int_0^0 G(t) - 100\ dt + 500 = 500$
4.923...	$\int_0^{4.923...} G(t) - 100\ dt + 500 = 635.376...$
8	$\int_0^8 G(t) - 100\ dt + 500 = 525.551...$

Since the largest value of these candidates is 635.376..., the maximum value of the function is $\boxed{\text{635.376... tons.}}$

5 Graphical Functions

Question 3 AP Calculus AB 2016

(NO CALCULATOR) The figure [below] shows the graph of the piecewise linear function f. For $-4 \leq x \leq 12$, the function g is defined by $g(x) = \int_2^x f(t)\ dt$.



Graph of f

Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \leq x \leq 12$. Justify your answers.

Nothing mathematically changes for graphical functions, but this example is here for clarity.

The interval is $[-4, 12]$, and the function is continuous on the interval so its integral, too, is continuous (but not differentiable!).

The relevant function is given. To obtain the critical points one equates the derivative equal to zero and solves.

$$\begin{aligned} g'(x) & &= 0 \\ g'(x) &= \frac{d}{dx}\int_2^x f(t)\ dt &= 0 \\ g'(x) &= f(x) &= 0 \end{aligned}$$

This yields the following solutions.

$$x = -2, 2, 6, 10$$

One can proceed with these values but, by inspection, points at $x = 2$ and $x = 10$ do not change sign while points $x = -2$ and $x = 6$ do. By the second derivative test, the first two cannot be extrema. Including the endpoints, the candidates are the following.

$$x = -4, -2, 6, 12$$

Then to identify the absolute minimum and maximum values on the interval, one evaluates the function at all of the candidates.

x	$g(x)$			
-4	$\int_2^{-4} f(t)\ dt$	$= \frac{1}{2}(-4)(-2) + \frac{1}{2}(4)(-2) + \frac{1}{2}(4)(-2)$	$=$	-4
-2	$\int_2^{-2} f(t)\ dt$	$= \frac{1}{2}(4)(-2) + \frac{1}{2}(4)(-2)$	$=$	-8
6	$\int_2^{6} f(t)\ dt$	$= \frac{1}{2}(4)(4)$	$=$	8
12	$\int_2^{12} f(t)\ dt$	$= \frac{1}{2}(4)(4) + \frac{1}{2}(4)(-4) + \frac{1}{2}(4)(-4)$	$=$	-4

Since the smallest value of these candidates is -8 and the largest of these values is 8, the absolute minimum and maximum of the function on the interval are $\boxed{\text{-8 and 8}}$ respectively.

6 Pathological Functions

The AP Calculus test has never used a discontinuous function on a question that requires finding an absolute extrema to my knowledge. However, this does not mean they will never show up. The process in doing them is mostly the same, with a few differences.

Plan B: Discontinuous Function

1. **Acknowledge interval and discern if the function diverges (goes to positive or negative infinity)**
   - If the function diverges in the positive/negative direction, there cannot be an absolute maximum/minimum on the interval.

2. **Generate a list of all critical points within the interval.**

3. **Plug in all critical points, the endpoints *and any points of discontinuity.* These are your so-called, "candidates."**

4. **Pick the smallest/largest value.**

7 Equation Examples

Question 2b/c AP Calculus AB 2002 FORM B

(CALCULATOR ACTIVE) The number of gallons, $P(t)$, of a pollutant in a lake changes at the rate $P'(t) = 1 - 3e^{-0.2\sqrt{t}}$ gallons per day, where t is measured in days. There are 50 gallons of the pollutant in the lake at time $t = 0$. The lake is considered to be safe when it contains 40 gallons or less of pollutant.

Is the lake safe when the number of gallons of pollutant is at its minimum? Justify your answer.

Question 1c AP Calculus AB 2002

(CALCULATOR ACTIVE) Let f and g be the functions given by $f(x) = e^x$ and $g(x) = \ln(x)$

Let h be the function given by $h(x) = f(x) - g(x)$. Find the absolute minimum value of $h(x)$ on the closed interval $\frac{1}{2} \leq x \leq 1$, and find the absolute maximum value of $h(x)$ on the closed interval $\frac{1}{2} \leq x \leq 1$. Show the analysis that leads to your answers.

Question 2d AP Calculus AB 2003 FORM B

(CALCULATOR ACTIVE) A tank contains 125 gallons of heating oil at time $t = 0$. During the time interval $0 \leq t \leq 12$ hours, heating oil is pumped into the tank at the rate

$$H(t) = 2 + \frac{10}{(1 + \ln(t + 1))} \text{ gallons per hour.}$$

During the same time interval, heating oil is removed from the tank at the rate

$$R(t) = 12 \sin\left(\frac{t^2}{47}\right) \text{ gallons per hour.}$$

At what time t for $0 \leq t \leq 12$, is the volume of heating oil in the tank the least? Show the analysis that leads to your conclusion.

Question 2d AP Calculus AB 2003

(CALCULATOR ACTIVE) A particle moves along the x-axis so that its velocity at time t is given by

$$v(t) = -(t+1)\sin\left(\frac{t^2}{2}\right)$$

At time $t = 0$, the particle is at position $x = 1$.

During the time interval $0 \leq t \leq 3$, what is the greatest distance between the particle and the origin? Show the work that leads to your answer.

Question 2c AP Calculus AB 2005 FORM B

(CALCULATOR ACTIVE) A water tank at Camp Newton holds 1200 gallons of water at time $t = 0$. During the time interval $0 \leq t \leq 18$ hours, water is pumped into the tank at the rate

$$W(t) = 95\sqrt{t}\sin^2\left(\frac{t}{6}\right) \text{ gallons per hour.}$$

During the same time interval, water is removed from the tank at the rate

$$R(t) = 275\sin^2\left(\frac{t}{3}\right) \text{ gallons per hour.}$$

At what time t, for $0 \leq t \leq 18$, is the amount of water in the tank at an absolute minimum? Show the work that leads to your conclusion.

Question 2d AP Calculus AB 2005

(CALCULATOR ACTIVE) The tide removes sand from Sandy Point Beach at a rate modeled by the function R, given by

$$R(t) = 2 + 5\sin\left(\frac{4\pi t}{25}\right).$$

A pumping station adds sand to the beach at a rate modeled by the function S, given by

$$S(t) = \frac{15t}{1+3t}.$$

Both $R(t)$ and $S(t)$ have units of cubic yards per hour and t is measured in hours for $0 \leq t \leq 6$. At time $t = 0$, the beach contains 2500 cubic yards of sand.

For $0 \leq t \leq 6$, at what time t is thea mount of sand on the beach a minimum? What is the minimum value? Justify your answers.

Question 2c AP Calculus AB 2007

(CALCULATOR ACTIVE) The amount of water in a storage tank, in gallons, is modeled by a continuous function on the time interval $0 \leq t \leq 7$, where t is measured in hours. In this model, rates are given as follows:
The rate at which water enters the tank is $f(t) = 100t^2 \sin(\sqrt{t})$ gallons per hour for $0 \leq t \leq 7$.
The rate at which water leaves the tank is

$$g(t) = \begin{cases} 250 & \text{for } 0 \leq t < 3 \\ 2000 & \text{for } 3 < t \leq 7 \end{cases} \text{ gallons per hour.}$$

For $0 \leq t \leq 7$, at which time t is the amount of water in the tank greatest? To the nearest gallon, compute the amount of water at this time. Justify your answer.

Question 2c AP Calculus AB 2007

(CALCULATOR ACTIVE) A storm washed away sand from a beach, causing the edge of the water to get closer to a nearby pond. The rate at which the distance between the road and the edge of the water was changing during the storm is modeled by $f(t) = \sqrt{t} + \cos t - 3$ meters per hour, t hours after the storm began. The edge of the water was 35 meters from the road when the storm began, and the storm lasted 5 hours. The derivative of $f(t)$ is $f'(t) = \frac{1}{2\sqrt{t}} - \sin t$.

At what time during the 5 hours of the storm was the distance between the road and the edge of the water decreasing most rapidly? Justify your answer.

Question 1c AP Calculus AB 2015

(CALCULATOR ACTIVE) The rate at which rainwater flows into a drainpipe is modeled by the function R, where $R(t) = 20\sin\left(\frac{t^2}{35}\right)$ cubic feet per hour, t is measured in hours, and $0 \leq t \leq 8$. The pipe is partially blocked, allowing water to drain out the other end of the pipe at a rate modeled by $D(t) = -0.04t^3 + 0.4t^2 + 0.96t$ cubic feet per hour, for $0 \leq t \leq 8$. There are 30 cubic feet of water in the pipe at time $t = 0$.

At what time t, $0 \leq t \leq 8$, is the amount of water in the pipe at a minimum? Justify your answer?

Question 5c AP Calculus AB 2018

(NO CALCULATOR) Let f be the function defined by $f(x) = e^x \cos(x)$.

Find the absolute minimum value of f on the interval $0 \leq x \leq 2\pi$. Justify your answer.

7.1 Graphical Function Examples

Due to difficulty in reproducing much of the graphs on the AP Calculus test, links will be provided in their stead.

These are links!

1. Question 5c AP Calculus AB 1999
2. Question 3d AP Calculus AB 2000
3. Question 3c AP Calculus AB 2001

4. Question 4b AP Calculus AB 2004 FORM B
5. Question 5c AP Calculus AB 2004
6. Question 2b AP Calculus AB 2006 FORM B
7. Question 4d AP Calculus AB 2007 FORM B
8. Question 5b AP Calculus AB 2008 FORM B
9. Question 4a AP Calculus AB 2008
10. Question 6c AP Calculus AB 2009
11. Question 4b AP Calculus AB 2010 FORM B
12. Question 3c AP Calculus AB 2010
13. Question 4b AP Calculus AB 2011
14. Question 4b AP Calculus AB 2013
15. Question 3c AP Calculus AB 2016
16. Question 3c AP Calculus AB 2017