Exploring Vieta's Formulas

Franklin High School — DeVon Herr

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1 Introduction

Various math competitions have problems that deal operations on roots. This can seem simple, such as adding all the roots of a polynomial or multiplying them. However, for more complicated, long polynomials, this task may seem daunting. This will get even more difficult with "weirder" operations on roots. We hope to make such problems far less difficult, and hopefully, easy. The concepts taught here are part of a competitive mathematician's toolkit; by far, these concepts are the most used *specific* technique in math competitions.

2 Motivations

You will be able to attack such problems such as the following:

- Let the roots of the polynomial $3x^2 + 18x + 27$ be r_1 and r_2, find both the sum and product of the roots.
- If a polynomial has a first degree coefficient of 34 and one root of the polynomial is 2, what is the other root?

And many more. Much harder ones too.

3 Testing the Waters

The following exercises are to help build intuition for some of the underlying mechanics of Vieta's Formulas. I recommend you do them, but are not necessary for comprehension.

1. Find the product of the (not necessarily distinct) roots of $x^2 + 2x + 1$.
2. Find the sum of the (not necessarily distinct) roots of $x^2 - 5x + 6$.
3. What are the sum and product of the (not necessarily distinct) roots of $x^2 + 4x - 21$?

3.1 Answers

1. The roots are -1 and -1, which multiply for a product of 1.
2. The roots are 2 and 3, which add for a sum of 5.
3. The roots are -7 and 3, which add and multiply for a sum and product of -4 and 21, respectively.

3.2 Our State Motto!

It's Eureka if you forgot or did not know. (*Authors Note: I found this out when I looked up how to spell it.*) Look at the answers and the questions. **The numbers match up!**
The first polynomial has terms x^2, $2x$ and 1, and the roots sum to 1, one of the coefficients.
The second polynomial's sum of 5 match (albeit a different sign) the $-5x$ term.
We see this twofold with the third polynomial; the sum matches the $4x$ term (though opposite sign), the product matches the -21 term.
Make a prediction that relates the sum and products of the roots of a polynomial to the coefficients. We will explore this in the next section.

4 Deriving Vieta's Formulas

4.1 Simple — Quadratics

Let's say we have a polynomial $x^2 + bx + c$, and we know it's (not necessarily distinct) roots to be p and q.
Note that the x^2 term's coefficient is 1. This is done for simplicity. However, you can convert any quadratic $ax^2 + bx + c$ into one of form $x^2 + bx + c$ by dividing all terms by a.
By the factor theorem, we know that the polynomial can be expressed as the product of it's factors, so $x^2 + bx + c = (x - p)(x - q)$.

By distributing the right hand side, we get
$x^2 + bx + c = x^2 - px - qx + pq$.

Grouping term wise and factoring the right hand side, we get
$x^2 + bx + c = x^2 - 1 \cdot p \cdot x - 1 \cdot q \cdot x + pq$.

Factoring completely, we get
$x^2 + bx + c = x^2 - (p + q)x + pq$.

Two polynomials can only be equal if and only if their coefficients are equal (and we know these two are!), so this means each pair of coefficients has to equal each other. More particularly, this means the following:
$bx = -(p + q)x$ or $b = -(p + q)$.
$c = pq$.
In English, this means that for quadratics, the product of the roots is given by the constant term/coefficient, and the sum of the roots is given by the first degree term's coefficient (the one next to the x), with it's sign flipped.

4.2 Practice Problems

If you really want to see how significant Vieta's formulas are, do these problems both ways, one with the classical method, and one with Vieta's method.

1. If α and β are the roots of the polynomial $x^2 - 4x + 9$, find the values of:
   (a) $\alpha + \beta$
   (b) $\alpha\beta$
   (c) $\alpha^2 + \beta^2$
   Hint! Expand $(\alpha + \beta)^2$ and manipulate from there.
2. Find a quadratic that has roots $3 + 2i$ and $3 - 2i$.
   AN: If you don't know how to multiply complex numbers, replace the question with $\sqrt{17}$ and $\sqrt{23}$

4.3 Solutions

1. (a) $\alpha + \beta = 4$
   (b) $\alpha\beta = 9$
   (c) $(\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$
   $(\alpha + \beta)^2 - 2\alpha\beta = \alpha^2 + 2\alpha\beta + \beta^2 - 2\alpha\beta$
   $(\alpha + \beta)^2 - 2\alpha\beta = \alpha^2 + \beta^2$
   AN: I swapped the sides here.

   $\alpha^2 + \beta^2 = (4)^2 - 2(9)$
   $\alpha^2 + \beta^2 = 16 - 18$
   $\alpha^2 + \beta^2 = -2$
2. We know the sum of the two roots, 6, will be the linear term with a flipped sign, and the product of the two roots, 13, will be the constant term. This gives us:
   $x^2 - 6x + 13$

4.4 Complicated — Cubics

We can derive a similar set of root properties for cubics. Although I recommend you do so on your own, I will do so here for completion and reference purposes.
Let the cubic polynomial $x^3 + bx^2 + cx + d$ have roots p, q, and r.

By the factor theorem, $x^3 + bx^2 + cx + d = (x - p)(x - q)(x - r)$

Distributing,
$x^3 + bx^2 + cx + d = x^3 - px^2 - qx^2 - rx^2 + pqx + prx + qrx - pqr$

Factoring,
$x^3 + bx^2 + cx + d = x^3 - (p + q + r)x^2 + (pq + pr + qr)x - pqr$

Do you see a pattern?
The signs alternate, and each term, going up from the end, has fewer roots being multiplied at a time. The constant term, $-pqr$, for example, has three roots being multiplied; the linear term, $(pq+pr+qr)x$, has a sum of two root products; the quadratic term, $-(p + q + r)x^2$, has a sum of only one root at a time.
If you care about jargon, the operations that make the coefficients have a name. If you sum the products of k numbers, it is known as the k-th elementary symmetric sum.

4.5 Complex — General Polynomials

The k-th elementary sum of the roots with (with sign alternation) is equal to the k-th coefficient.

4.6 Practice Problems

Some of these problems are difficult and take a lot of time, so the solutions will be somewhere else (the mathletes folder, presumably).

1. Let r_1, r_2 and r_3 be the three roots of the cubic x^3+3x^2+4x-4. Find the value of $r_1r_2+r_1r_3+r_2r_3$
2. Suppose the polynomial $5x^3 + 4x^2 - 8x + 6$ has three roots a, b, and c. Find the value of $a(1 + b + c) + b(1 + a + c) + c(1 + a + b)$
3. Let a, b, and c be positive real numbers (with the condition $a < b < c$) such that $a + b + c = 12$, $a^2 + b^2 + c^2 = 50$, and $a^3 + b^3 + c^3 = 216$. Find $a + 2b + 3c$.
4. Find $c > 0$ such that if r, s, and t are the roots of the cubic $f(x) = x^3 - 4x^2 + 6x - c$, then $1 = \frac{1}{r^2+s^2} + \frac{1}{s^2+t^2} + \frac{1}{t^2+r^2}$

5 Conclusion

Thank you for reading! I hope you've learned, and hopefully will remember, the information covered in this document. Let me know any suggestions/criticisms you may have.