A Sample Optimization Problem

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1 Introduction

Optimization is a pretty broad topic in calculus, with most questions taking the form of a word problem generally relating to geometry. That being said, there's not much deep "theory" in regards to doing the questions. In essence, there's one goal in order to solve the questions.

"Set relevant derivative equal to zero"

2 The Example!

An open-topped glass aquarium with a square base is designed to hold 32 cubic feet of water. What is the minimum possible exterior surface area of the aquarium?

3 Step 1: Identify Constraint

There's no rule you have to follow in the order of your steps, but generally finding the constraint first is better for two reasons. One, it's usually easier. The other is because by gleaming at the constraint, one can kind of gauge what to look for when solving the question.

The aquarium needs to perfectly hold 32 cubic feet of water. More directly speaking, the volume of aquarium has to be 32 cubic feet.

$$V = 32 \text{ cubic feet}$$

4 Step 2: Transform Constraint

This requires you to identify the object in the question. We will make the assumption that the aquarium is nice and well-behaved, and that the walls are parallelograms (like, an actual rectangular prism normal aquarium).

We want to transform our condition that the volume of our aquarium is 32 cubic feet into something more "direct."

What is the volume of our aquarium? Recall the formula for a rectangular prism.

$$\text{Volume} = \text{Length} \cdot \text{Width} \cdot \text{Height}$$

The question explicitly states that the aquarium has a square base, so we have to use this condition to simplify our statement. In particular, if the base of a rectangular prism is square, the lengths and widths (or any two dimensions) are the same. This simplifies our volume equation.

$$\text{Length} = \text{Width} \Rightarrow \text{Volume} = \text{Length}^2 \cdot \text{Height}$$

Since we know that the volume of the aquarium is 32 cubic feet, we can substitute this into the constraint equation.

$$\text{Volume} = 32 \text{ cubic feet} \Rightarrow 32 \text{ cubic feet} = \boxed{\text{Length}^2 \cdot \text{Height}}$$

We then need to solve for either variable. Solving for x is bad since you have to take a square root which messes things up.

$$\boxed{\text{Height} = \frac{32 \text{ cubic feet}}{\text{Length}^2}}$$

Okay, we should keep in mind that we know have a relationship between the aquariums volume, height and length.

5 Step 3: Find Relevant Function

The question asks for the surface area of an open-topped glass aquarium with a square base. Since the surface area of an object is given by adding together the areas of all of its sides, the surface area is informally:

$$\text{Surface Area} = \text{Area(Base)} + \text{Area(Wall}_1) + ... + \text{Area(Wall}_4)$$

Since the walls are parallelograms, each wall should have the same area. This simplifies to the following.

$$\boxed{\text{Surface Area} = \text{Area(Base)} + 4 \cdot \text{Area(Wall)}}$$

6 Step 4: Transform Relevant Function

Since we have a relationship between volume, length and height, we should try to substitute in those values into the equations wherever we can. Since the area of a square is just the square of its sides, which in this case is the length of the aquarium (or width), the area of the square is transformed.

$$\text{Area(Base)} = \text{Length}^2 \Rightarrow \text{Surface Area} = \boxed{\text{Length}^2 + 4 \cdot \text{Area(Wall)}}$$

We repeat this procedure with the area of the walls. The area of the walls is given by the product of the length (or width) of the aquarium multiplied by the height.

$$\text{Area(Wall)} = \text{Length} \cdot \text{Height} \Rightarrow \text{Surface Area} = \text{Length}^2 + 4(\text{Length} \cdot \text{Height})$$

There's three variables here (surface area, length and height) which is just a mess. We will look to use some sort of substitution to reduce this. Recall that $\text{Height} = \frac{\text{32 cubic feet}}{\text{Length}^2}$. Substituting into the function does indeed simplify it.

$$4(\text{Length} \cdot \text{Height}) = 4 \cdot \left(\text{Length} \cdot \frac{\text{32 cubic feet}}{\text{Length}^2}\right) \Rightarrow \boxed{\frac{\text{128 cubic feet}}{\text{Length}}}$$

Putting it all together generates the following relevant equation.

$$\boxed{\text{Surface Area} = \text{Length}^2 + \frac{\text{128 cubic feet}}{\text{Length}}}$$

7 Step 5: Optimize Relevant Function

Since optimization questions ask for a minimal or maximal value, it's natural to rephrase the question as asking to minimize/maximize of a function.

Our relevant function is the following.

$$\text{Surface Area} = \text{Length}^2 + \frac{\text{128 cubic feet}}{\text{Length}}$$

Differentiating in terms of length gives the following.

$$\boxed{\text{Surface Area}' = 2 \cdot \text{Length} - \frac{\text{128 cubic feet}}{\text{Length}^2}}$$

The minimal/maximal values lie at the critical points. These will be the endpoints and where the derivative equals zero. Due to some weird stuff (multiplying by zero or dividing by zero), we will ignore the endpoints here. More generally, the endpoints tend to also not show up very often.

$$
\begin{aligned}
2 \cdot \text{Length} - \frac{128 \text{ cubic feet}}{\text{Length}^2} &= 0 \\
2 \cdot \text{Length}^3 - 128 \text{ cubic feet} &= 0 \\
\text{Length}^3 - 64 \text{ cubic feet} &= 0 \\
(\text{Length} - 4 \text{ feet})(\text{Length}^2 + \text{Length} \cdot 4 \text{ feet} + 16 \text{ square feet}) &= 0 \\
\text{Length} - 4 \text{ cubic feet} &= 0 \\
\therefore &\boxed{\text{Length} = 4}
\end{aligned}
$$

Therefore the maximal value occurs when the length of the container is 4. Evaluating the surface area function at this value gives the maximal surface area. Recalling the surface area equation and evaluating gives the following.

$$
\begin{aligned}
\text{Surface Area}\Big|_{\text{Length}=4 \text{ feet}} &= \text{Length}^2 + \frac{128 \text{ cubic feet}}{\text{Length}}\Bigg|_{\text{Length}=4} \\
&= (4 \text{ feet})^2 + \frac{128 \text{ cubic feet}}{4 \text{ feet}} \\
&= 16 \text{ square feet} + 32 \text{ square feet} \\
&= \boxed{48 \text{ square feet}}
\end{aligned}
$$

8 Step 6: Verify Solution

This is to be done before evaluating the function in the case of multiple critical values.

Differentiate the derivative to get the second derivative of the relevant function, then evaluate to see if it is concave up or down, which can then discern if the critical point is minimal or maximal.

$$
\begin{aligned}
\text{Surface Area}''\Big|_{\text{Length}=4 \text{ feet}} &= \left(2 \cdot \text{Length} - \frac{128 \text{ cubic feet}}{\text{Length}^2}\right)'\Bigg|_{\text{Length}=4 \text{ feet}} \\
&= 2 + \frac{256 \text{ cubic feet}}{\text{Length}^3}\Bigg|_{\text{Length}=4 \text{ feet}} \\
&= 2 + \frac{256 \text{ cubic feet}}{(4 \text{ feet})^3} \\
&= 2 + 4 \\
&= 6 > 0
\end{aligned}
$$

Since the second derivative is greater than 0, by the second derivative test, the function is concave up when evaluated at Length = 6. 48 square feet is a minimum, and we are done.